

4.1. Polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$$

$\downarrow$ initial coefficient $\downarrow$ constant term

$n =$ degree of polynomial
(order)

non-negative integers only

$$f(x) = 5 - \frac{2}{x^2}$$

$$5 - 2x^{-2}$$

Not a polynomial

$$g(x) = \underbrace{5 - 2x^2}_{\text{2nd degree polynomial}}$$

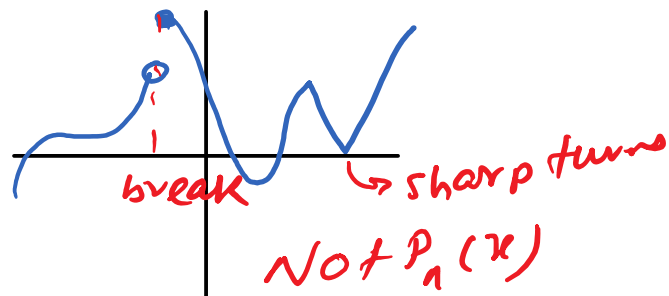
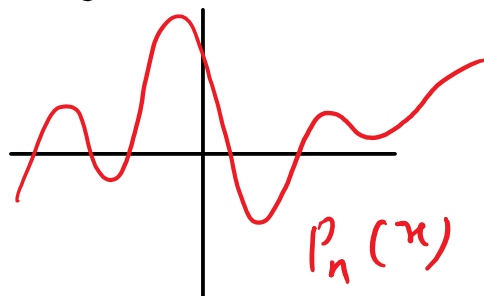
2nd degree polynomial.

$$P_1(x) \rightarrow 1^{\text{st}} \text{ degree poly: } y = mx + b$$

$$P_2(x) \rightarrow 2^{\text{nd}} \quad \quad \quad y = ax^2 + bx + c$$

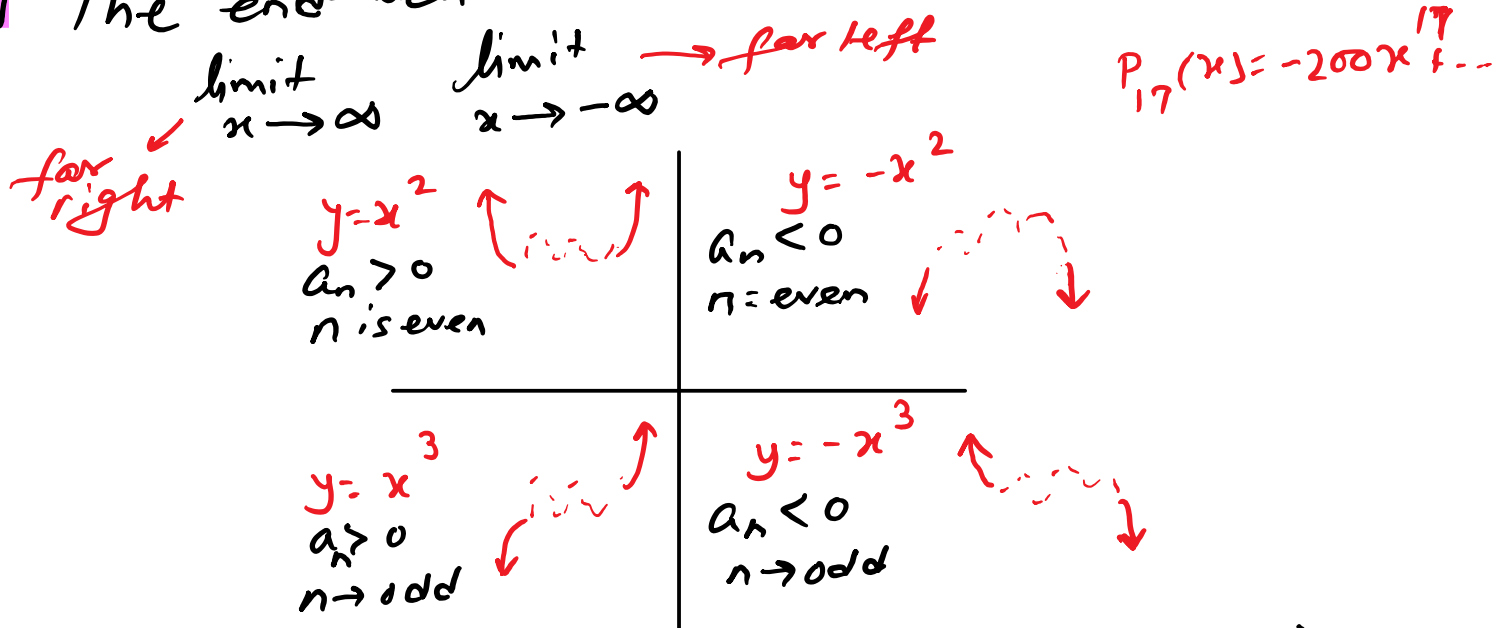
$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$\rightarrow$ graphs are continuous and smooth



$\rightarrow (n-1) =$ maximum number of turns

□ The end-behavior

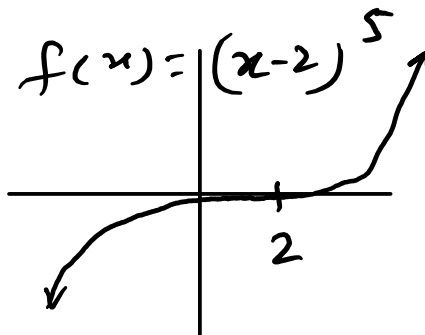
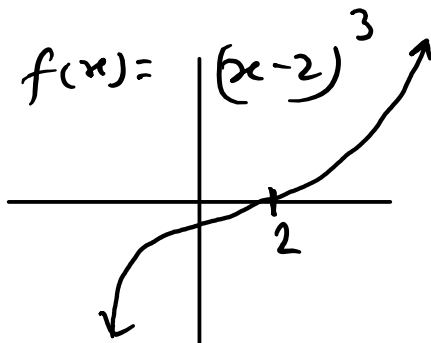


□ Behavior at x -intercept (behavior at zero)

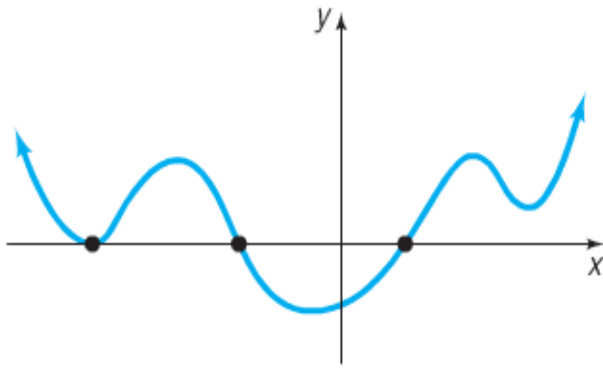
$f(x) = (x-2)^3 \rightarrow (x-2)^3 = 0$
 $x-2 = 0$
 $x = 2$
(zero)

- 3 → multiplicity of zero
 □ crosses x -axis at $x=2$
 □ higher multiplicity = flattens at zero

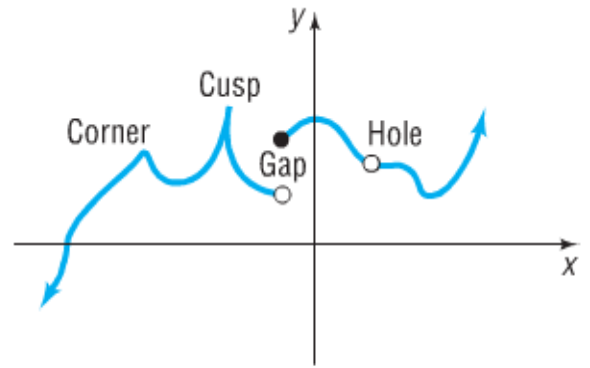
$g(x) = (x-2)^4 \rightarrow \left. \begin{array}{l} \text{zero: } x=2 \\ \text{multiplicity}=4 \end{array} \right\} \begin{array}{l} \square \text{ touches} \\ x\text{-axis} \\ \text{does not} \\ \text{cross} \end{array}$



4.1p3

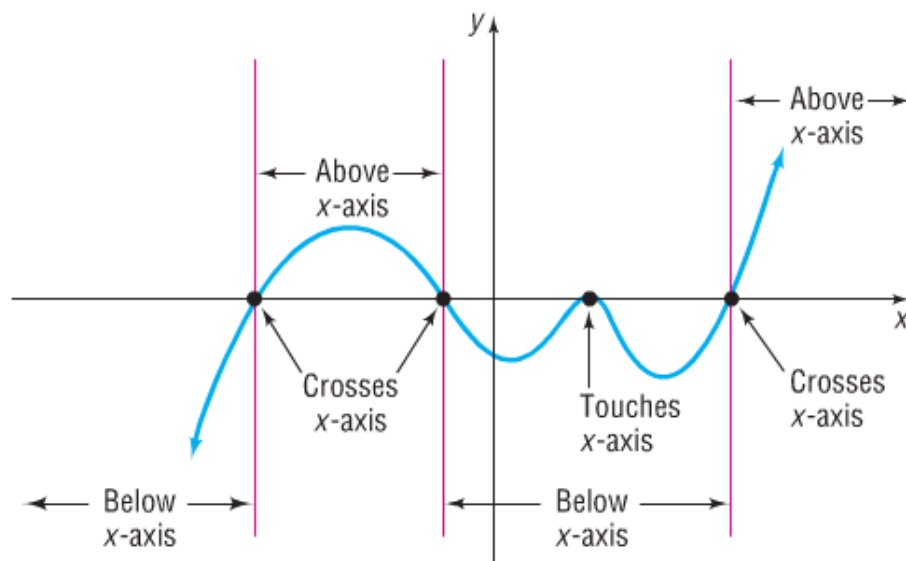


(a) Graph of a polynomial function:
smooth, continuous



(b) Cannot be the graph of a
polynomial function

Zero-factor, multiplicity(4.1p4)



If $x=a$ is a zero, then $(x-a)$ is a factor of the polynomial

A polynomial can be written as the product of its factors.

$$P(x) = A (x-a)^m (x-b)^l (x-c)^p \dots$$

for vertical stretch

$x=a$ is zero

$x=b$ is a zero

$x=c$ is a zero

m → multiplicity of the zero

l

p

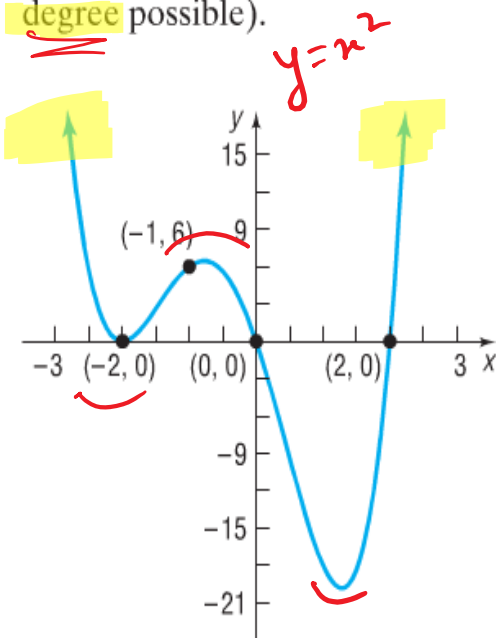
- If multiplicity is even, graph only touches x -axis, does not cross-over
- If multiplicity is odd, graph crosses x -axis.

Behavior at x -intercept

4/1p5Graph to function

Writing a Polynomial Function from Its Graph

Write a polynomial function whose graph is shown in Figure 14 (use the smallest degree possible).



of turns = 3

$n = (4) 5, 6, \dots$

$P_4(x)$



A polynomial of degree n , cannot turn more than $(n-1)$ times

zeros

multiplicity

factors

$x = -2$

2

$(x+2)^2$
 $(x+2)^2 = 0$

$x+2=0$
 $x=-2$, mult=2

$x = 0$

1

$(x-0)^1$

$x = 2$

1

$(x-2)^1$

$$P_4(x) = A (x+2)^2 (x-0)^1 (x-2)^1$$

$$P_4(x) = A (x+2)^2 (x)(x-2)$$

$$\downarrow (-1, 6) \quad \downarrow \quad \downarrow \quad \downarrow$$

$$6 = A (-1+2)^2 (-1) (-1-2)$$

$$6 = A (1) (-1) (-3) \rightarrow 6 = 3A \rightarrow A = 2$$

$$P_4(x) = 2 (x+2)^2 (x)(x-2)$$

$6 = 3 \cdot 2 \cdot 1$
factor factor factor

$$6 = A(1)(-1)(-3) \rightarrow 6 = \dots$$

$$\boxed{p_4(x) = 2(x+2)^2(x)(x-2)}$$

$\underbrace{\quad\quad\quad}_{2} \quad \underbrace{\quad}_{1} \quad \underbrace{\quad}_{1} = n=4$

4.1#51-56

In Problems 51–56, find the polynomial function with the given zeros whose graph passes through the given point.

51. Zeros: $-3, 1, 4$
Point: $(6, 180)$

52. Zeros: $-2, 0, 2$
Point: $(-4, 16)$

53. Zeros: $-1, 0, 2, 4$
Point: $\left(\frac{1}{2}, 63\right)$

54. Zeros: $-5, -1, 2, 6$
Point: $\left(\frac{5}{2}, 15\right)$

55. Zeros: -1 (multiplicity 2),
 1 (multiplicity 2)
Point: $(-2, 45)$

56. Zeros: 0 (multiplicity 1),
 $-1, 3$ (multiplicity 2)
Point: $(1, -48)$

zero mult factors

zero mult. factor
 $x=0$ 1 $(x-0)=x$

$x=-1$ 1 $(x+1)$

$x=3$ 2 $(x-3)^2$

$$P_4(x) = A(x)(x+1)(x-3)^2$$

$(1, -48)$

$$-48 = A(1)(2)(-2)^2$$

$$-48 = 8A$$

$$-6 = A$$

$$P_4(x) = -6(x)(x+1)(x-3)^2$$

4.1#43-50

In Problems 43–50, form a polynomial function whose real zeros and degree are given. Answers will vary depending on the choice of a leading coefficient.

43. Zeros: $-1, 1, 3$; degree 344. Zeros: $-2, 2, 3$; degree 345. Zeros: $-3, 0, 4$; degree 346. Zeros: $-4, 0, 2$; degree 347. Zeros: $-4, -1, 2, 3$; degree 448. Zeros: $-3, -1, 2, 5$; degree 449. Zeros: -1 , multiplicity 1; 3 , multiplicity 2; degree 350. Zeros: -2 , multiplicity 2; 4 , multiplicity 1; degree 3

<u>50) zeros</u>	<u>mult</u>	<u>factors</u>
$x = -2$	2	$(x+2)^2$
$x = 4$	1	$(x-4)^1$

$$\begin{aligned}
 P_3(x) &= \square (x+2)^2 (x-4) \\
 &= \square (x^2 + 4x + 4)(x-4) \\
 &= \square (x^3 - 4x^2 + 4x^2 - 16x + 4x - 16)
 \end{aligned}$$

$$P_3(x) = \square (x^3 - 12x - 16)$$

4.1p(NEW)

Combination:

Sketch the graph of the polynomial

$$P_5(x) = -2(x+2)^3(x-1)^2$$

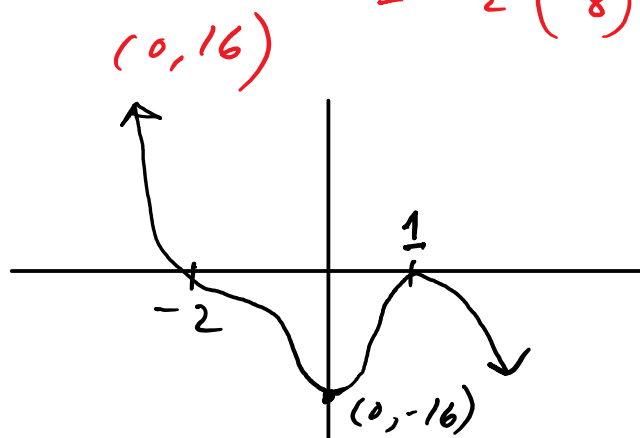
end behavior → initial coeff = $-2 < 0$; highest power of x
Degree = 5 (odd)

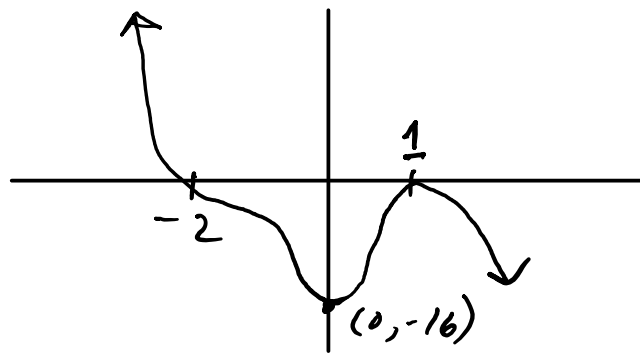
so, like $y = -x^3$

→ max. number of turns: $n-1 = 5-1 = \underline{4}$

Behavior at zero:	factor	Zero	multiplicity	
	$x+2=0$	$x=-2$	3	→ crosses
	$x-1=0$	$x=1$	2	→ Does not cross

→ y-int: $x=0 \rightarrow y = -2(0+2)^3(0-1)^2$
 $= -2(8)(1) = -16$





Number system

A review of Complex Number System.

All Numbers

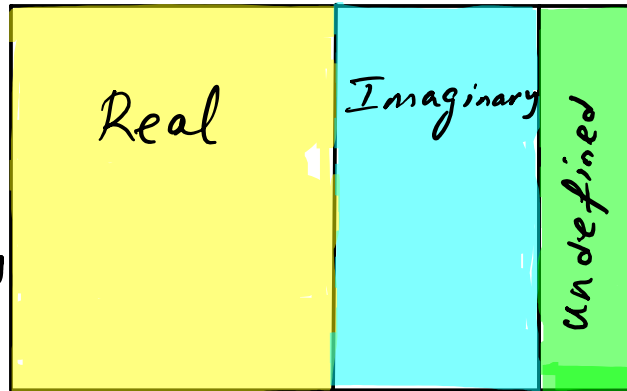
Complex Numbers

$$= A \pm Bi$$

Real part Imaginary part

If $A = 0; B \neq 0 \rightarrow$ Imaginary

If $A \neq 0; B = 0 \rightarrow$ Real



The Real Numbers:

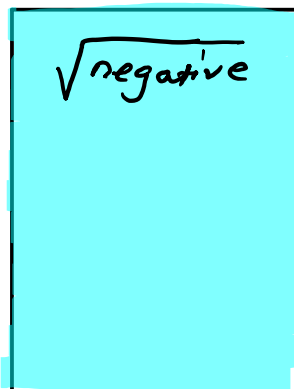
Rational	Irrational
$\frac{a}{b}, b \neq 0$	non-terminating
$\rightarrow$ integers	non-repeating
$\rightarrow$ whole	decimals
$\rightarrow$ Terminating decimals	$\sqrt{2}, \pi,$
$\rightarrow$ repeating decimals	$3 \pm \sqrt{17}$
	$\rightarrow$ cannot

be written as $\frac{a}{b}$

The Imaginary Numbers

$$\text{let } i^2 = -1$$

$$\text{so, } \sqrt{-4} = \sqrt{4(-1)} = \sqrt{4i^2} = \pm 2i$$



$$\rightarrow \sqrt{-24} = \sqrt{24i^2} = \sqrt{4(6)i^2} = \pm 2i\sqrt{6}$$

- 4.2 The Fundamental Theorem of Algebra
A polynomial $P_n(x)$, of degree n , MUST have exactly n - complex zeros

$P_3(x) \rightarrow 3$ complex zeros
1 real; 2 imaginary
3 reals

- Imaginary zeros and Irrational zeros
 $A + Bi, A - Bi$ $A + \sqrt{B}; A - \sqrt{B}$
always occur in conjugate pairs
 $(a+b)(a-b)$

- Zero - Factor theorem:
If $x = a$ is zero of $y = f(x)$, then $f(a) = 0$
 $\rightarrow f(b) \neq 0 \rightarrow b$ is not a zero. $(x-a)$ is a factor

Rational Zeros Theorem

Let f be a polynomial function of degree 1 or higher of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad a_n \neq 0, \quad a_0 \neq 0$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f , then p must be a factor of a_0 and q must be a factor of a_n .

p = all factors of a_0

q = all factors of a_n

all rational zeros belong to $= \pm \left(\frac{\text{all factors of } a_0}{\text{all factors of } a_n} \right)$

Division of polynomials

$$\begin{array}{r} 57 \\ 10 \end{array}$$

dividend
Divisor

$$\begin{array}{r} 5 \\ 10 \overline{) 57} \\ \underline{50} \\ 7 \end{array}$$

Quotient
Remainder

$$\frac{57}{10} = Q + \frac{R}{\text{Divisor}} = 5 + \frac{7}{10}$$

Long Division of Polynomials

$$\frac{P_3(x)}{P_1(x)}$$

$$\frac{x^3 - 3x + 5}{x - 2}$$

$$= Q + \frac{R}{D} = x^2 + 2x + 1 + \frac{7}{x-2}$$

$$\begin{array}{r} x^2 + 2x + 1 \\ x-2 \overline{) x^3 + 0x^2 - 3x + 5} \\ \underline{-x^3 + 2x^2} \\ 2x^2 - 3x + 5 \\ \underline{-2x^2 + 4x} \\ 1x + 5 \\ \underline{-x + 2} \\ 7 \end{array}$$

Quotient
Remainder

Synthetic Division

we only use if the divisor is linear ($ax+b$)

$$\frac{x^3 + 0x^2 - 3x + 5}{x - 2}$$

① $x - 2 = 0 \rightarrow x = 2$

② $x = 2$

	x^3	x^2	x	C
2	0	-3	5	
	2	4	2	
	1	2	1	7

Remainder

Reduced Quadratic

$$1 \quad 2 \quad 1$$

$$1x^2 + 2x + 1$$

$$x^2 + 2x + 1 + \frac{7}{x-2}$$

1 2 1 7 → Remainder !

4.2 / 4.3
 ↓
 real zeros complex zeros

#57) 4.2
 $f(x) = 1x^3 - 10x^2 + 28x - 16$
 Find all real zeros (complex zeros)

① Rational zeros = $\pm \left(\frac{1, 2, 4, 8, 16}{1} \right) \frac{P}{Q} = \pm (1, 2, 4, 8, 16)$

② get one rational zero from calc : $x = 4$

③ Synthetic Division → Reduced Quadratic

$$\begin{array}{r|rrrr}
 x=4 & 1 & -10 & 28 & -16 \\
 & & 4 & -24 & 16 \\
 \hline
 & 1 & -6 & 4 & 0=R
 \end{array}$$

$1x^2 - 6x + 4 = 0 \rightarrow \begin{matrix} a=1 \\ b=-6 \\ c=4 \end{matrix}$

④ Solve: $x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)} = \frac{+6 \pm \sqrt{36-16}}{2}$

$(0.76..., 0) \approx 3 - \sqrt{5}$

$(5.24, 0) \approx 3 + \sqrt{5}$

$3 \pm \sqrt{5} = \frac{6 \pm 2\sqrt{5}}{2} = \frac{6 \pm \sqrt{20}}{2}$

$\boxed{x = 4, 3 + \sqrt{5}, 3 - \sqrt{5}}$

4.2, 4.3p2

From given zeros $\rightarrow$ find $P_3(x)$

$$x = 4 \rightarrow (x - 4)$$

$$x = 3 - \sqrt{5} \rightarrow [x - (3 - \sqrt{5})] \rightarrow \left[\frac{x - 3 + \sqrt{5}}{(a+b)} \right] \left[\frac{x - 3 - \sqrt{5}}{(a-b)} \right] \rightarrow (a-b) = a^2 - b^2$$

$$x = 3 + \sqrt{5} \rightarrow [x - (3 + \sqrt{5})] \rightarrow (x - 3)^2 - (\sqrt{5})^2$$

$$x^2 - 6x + 9 - 5$$

$$x^2 - 6x + 4$$

$$(x - 4)(x^2 - 6x + 4)$$

$$= x^3 - 6x^2 + 4x - 4x^2 + 24x - 16$$

$$P_3(x) = x^3 - 10x^2 + 28x - 16$$

$$P_3(x) = (x - 4)(x^2 - 6x + 4)$$

$$(x - 4)(x - 3 + \sqrt{5})(x - 3 - \sqrt{5})$$

4.3

$$39) f(x) = 3x^4 - x^3 - 9x^2 + 159x - 52$$

$$\frac{13}{4} \\ \frac{52}{52}$$

Find all complex zeros.

① Rational zeros: $\frac{p}{q} = \pm \left(\frac{1, 2, 4, 13, 26, 52}{1, 3} \right)$
 $= \pm \left(1, 2, 4, 13, 26, 52, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{13}{3}, \frac{26}{3}, \frac{52}{3} \right)$

② $x = -4; x = \frac{1}{3}$

$$\begin{array}{r|rrrrr} x = -4 & 3 & -1 & -9 & 159 & -52 \\ & & -12 & 52 & -172 & 52 \\ \hline & 3 & -13 & 43 & -13 & 0=R \end{array}$$

$$\begin{array}{r|rrrr} x = \frac{1}{3} & 3 & -13 & 43 & -13 \\ & & 1 & -4 & 13 \\ \hline & 3 & -12 & 39 & 0=R \end{array}$$

Red Quad: $3x^2 - 12x + 39 = 0$

③ $x = \frac{12 \pm \sqrt{144 - 4(3)(39)}}{2(3)} = \frac{12 \pm \sqrt{144 - 468}}{6}$

$$\begin{array}{l} 324 \\ \swarrow \searrow \\ (2) \quad 162 \\ \swarrow \searrow \\ (2) \quad (81) \end{array}$$

$$4 \times 81$$

Ans:

$$x = \frac{12 \pm \sqrt{-324}}{6} = \frac{12 \pm \sqrt{324}i^2}{6} = \frac{12 \pm 18i}{6}$$

$$x = 2 \pm 3i$$

$$x = -4; \frac{1}{3}, 2+3i; 2-3i$$

7781

Ans:

$$\boxed{x = -4j \frac{1}{3}, 2+3j; 2-3j}$$

4.2(real zeros), 4.3(complex zeros)

4.3, 4.2

$$f(x) = 2x^3 + 11x^2 - 7x - 6$$

Find all zeros (real and imaginary) of $f(x)$

By FTA: 3 complex zeros.

By RZT: Rational zeros $\frac{p}{q} = \pm \left(\frac{1, 2, 3, 6}{1, 2} \right) = \pm (1, 2, 3, 6, \frac{1}{2}, \frac{3}{2})$

→ $x = -6$ is a zero.

→ Synthetic Division

$$\begin{array}{r|rrrr} x = -6 & 2 & 11 & -7 & -6 \\ & & -12 & 6 & +6 \\ \hline & 2 & -1 & -1 & 0 = R \end{array}$$

→ Reduces quadratic:

$$2x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(-1)}}{2(2)} = \frac{1 \pm \sqrt{1+8}}{4}$$

$$= \frac{1 \pm 3}{4}$$

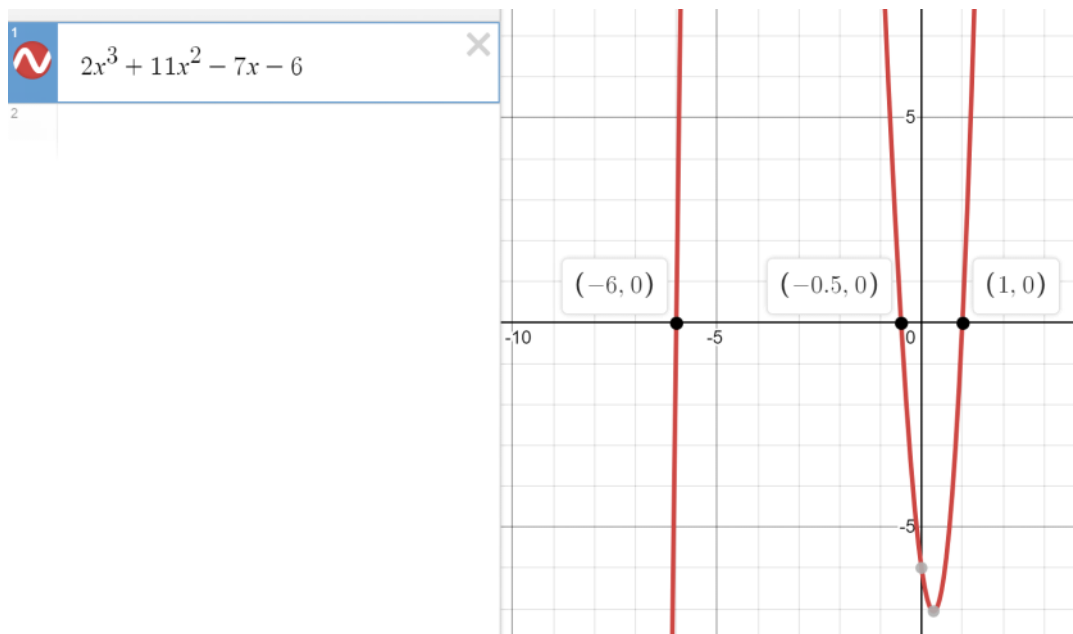
↙ ↘

1 $-\frac{1}{2}$

$$x = 1, -\frac{1}{2}, -6 \rightarrow \text{all zeros}$$

$$\begin{aligned} P_3(x) &= A \left[(x-1) \left(x + \frac{1}{2} \right) (x+6) \right] \\ &= A \left[(x-1) (2x+1) (x+6) \right] \end{aligned}$$

Graphical check



#79) $f(x) = 3x^3 - x^2 - 15x + 5$

find all zeros.

① $\frac{P}{Q}$

② calc

③ synth. Div $\rightarrow$ reduced quad

④ solve

① Rational: $\frac{P}{Q} = \pm \left(\frac{1, 5}{1, 3} \right) = \pm (1, 5, \checkmark 1/3, 5/3)$

② $x = \frac{1}{3}$

③ $x = \frac{1}{3}$

3	-1	-15	5
	1	0	-5
3	0	-15	0 = R

RQ: $3x^2 + 0x - 15 = 0$

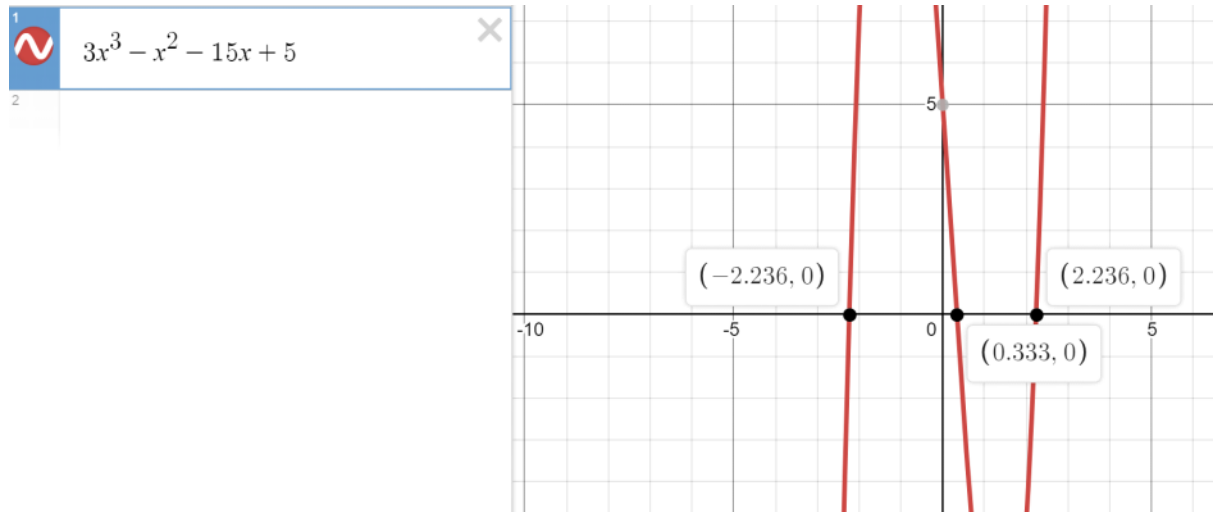
$3(x^2 - 5) = 0$

$x = \pm \sqrt{5}$

All zeros: $x = \frac{1}{3}, \sqrt{5}, -\sqrt{5}$

$P_3(x) = 3(x - \frac{1}{3})(x^2 - 5)$

Graphical check



4.3 example

4.3 Find all complex zeros:

$$f(x) = 3x^4 + 5x^3 + 25x^2 + 45x + 18$$

① Rational zeros: $\frac{p}{q} = \pm \left(\frac{1, 2, 3, 6, 9, 18}{1, 3} \right) = \pm \left(\frac{1}{3}, \frac{2}{3} \right)$

② get 2 rational zeros:

$$x = -2, \frac{1}{3}$$

③ $x = -2 \Big| \begin{array}{r} 3 \quad 5 \quad 25 \quad 45 \quad -18 \\ -6 \quad 2 \quad -54 \quad 18 \\ \hline 3 \quad -1 \quad 27 \quad -9 \quad 0=R \end{array}$

$x = \frac{1}{3} \Big| \begin{array}{r} 3 \quad -1 \quad 27 \quad -9 \\ 1 \quad 0 \quad 9 \\ \hline 3 \quad 0 \quad 27 \quad 0=R \end{array}$

④ Reduced quad:

$$3x^2 + 27 = 0$$

$$3(x^2 + 9) = 0$$

$$x^2 = -9$$

$$x = \pm \sqrt{-9} = \pm \sqrt{9i^2}$$

$$= \pm 3i \quad 2 \text{ imaginary}$$

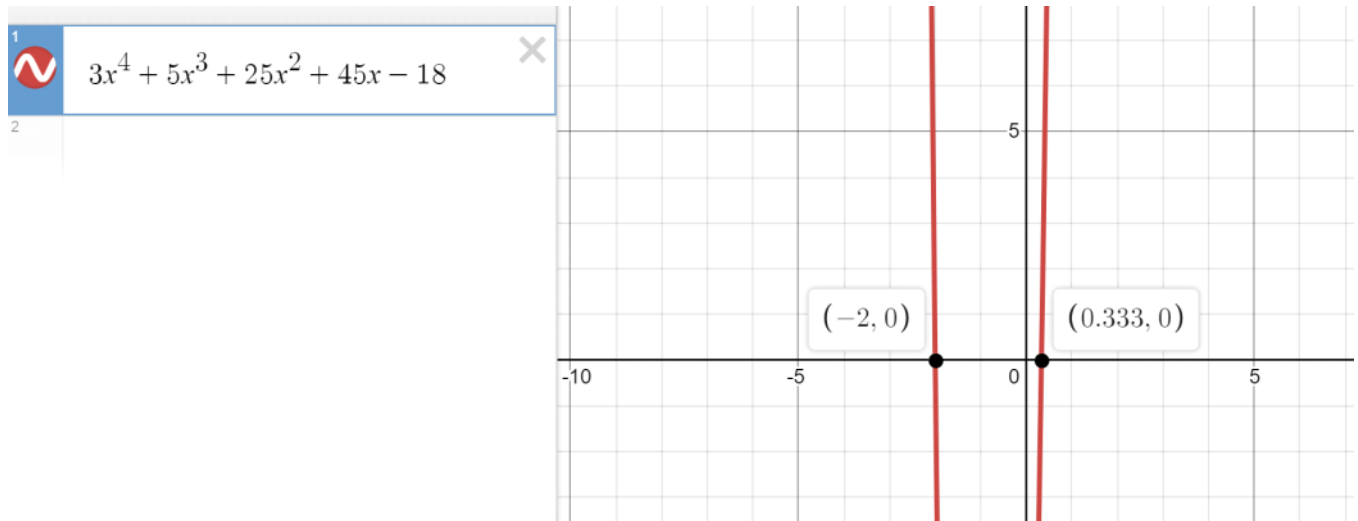
Zeros:

$$x = -2, \frac{1}{3}, +3i, -3i$$

$$P_4(x) = 3(x+2)\left(x-\frac{1}{3}\right)(x^2+9)$$

$$\begin{aligned} \sqrt{-5} &= \sqrt{5(-1)} \cdot i^2 = -1 \\ &= \sqrt{5i^2} \\ &= \pm i\sqrt{5} \end{aligned}$$

Graphical check



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